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ON THE UNKNOWN CONTRIBUTION OF STEFAN CHANAS TO THE STOCHASTIC FLOW SHOP ANALYSIS

The paper deals with the problem of minimizing the makespan in flow shops with random job processing times. We present recent advances in stochastic flow shop analysis in which the so-called reversibility property plays a crucial role. The paper is written in memory of Stefan Chanas, who observed first that this property must be utilized in order to tackle the complexity of stochastic flow shops.

1. Introduction

Several years ago when I visited Stefan Chanas, my long time friend and actual supervisor of my Ph.D. thesis, we discussed the classical problem of minimizing the makespan in two-machine flow shops. I raised then the question why pairwise adjacent job interchange arguments are much more difficult to follow in the stochastic case than in the deterministic case. Clearly, suppose a set of jobs has to be processed in a flow shop with two machines A and B. Let $\pi_{ij} = (\rho, i, j, \omega)$ and $\pi_{ji} = (\rho, j, i, \omega)$ be two job sequences, where ρ and ω are subsequences of jobs excluding i and j . Denote the makespans of π_{ij} and π_{ji} by M_{ij} and M_{ji} , and let D_{ij} and D_{ji} be the makespans of (ρ, i, j) and (ρ, j, i) , respectively. Then for the deterministic job processing times we have the following very intuitive relation: $D_{ij} \leq D_{ji}$ implies $M_{ij} \leq M_{ji}$. However, when the job processing times are independent random variables: $E(D_{ij}) \leq E(D_{ji})$ does not imply $E(M_{ij}) \leq E(M_{ji})$. When I presented the above observation to Stefan, he responded, "It looks like the jobs of ω must play an important role". After few seconds he added, "Why do we have to assume that the jobs are first processed on machine A and next on machine B. If we process the jobs in the reversed order and in the opposite direction, that is, first on B and next on A, the makespan remains the same."

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When later I reexamined the stochastic flow shop problem, I recalled Stefan's idea. This simple excellent idea led to several publications, including [5, 6, 7], in which Stefan's contribution remained unknown. Stefan strongly declined to have his name mentioned in acknowledgements. He pointed out that the reversibility property of flow shops had to have been known earlier, and as always he was right; see, e.g., Pinedo [9, p. 133]. However, this property was just stated and had never been utilized in the literature. The purpose of this paper is to summarize recent advances in stochastic flow shop analysis in which the reversibility property plays a crucial role.

2. Two-machine flow shops

A set of jobs, $\{1, 2, \dots, n\}$, available at time zero has to be processed in a shop with two machines A and B. Each job is processed first on A and next on B. No machine can process more than one job at a time, no preemption is allowed, all setup times are included into the job processing times, and there is unlimited intermediate storage in between the machines. The problem is to determine a job sequence (permutation) that minimizes the completion time of the last job, also known as the makespan.

2.1. Makespan representations

Let A_k and B_k denote the processing times of job k on machine A and B, respectively. To follow pairwise adjacent job interchange arguments, let $\pi_{ij} = (\rho, i, j, \omega)$ and $\pi_{ji} = (\rho, j, i, \omega)$ be two job sequences, where ρ and ω are subsequences of jobs excluding i and j , and let M_{ij} and M_{ji} be the makespans of π_{ij} and π_{ji} . As Stefan noticed, if the jobs of a given sequence are processed in the reverse order and in the opposite direction (first on B and next on A), the makespans of both sequences are the same. By the reverse sequence of a given job sequence we mean the sequence defined above. Denote the sequences reverse to π_{ij} and π_{ji} by $\underline{\pi}_{ij}$ and $\underline{\pi}_{ji}$, respectively. Thus, M_{ij} and M_{ji} are also the makespans of $\underline{\pi}_{ij}$ and $\underline{\pi}_{ji}$. We assume that the jobs are processed as soon as possible (without unnecessary idle times) under π_{ij} , $\underline{\pi}_{ij}$, π_{ji} , and $\underline{\pi}_{ji}$.

The following theorem gives convenient representations of M_{ij} and M_{ji} ; compare with Theorem 1 in [6].

Theorem 1. *The makespans M_{ij} and M_{ji} can be represented as follows:*

$$M_{ij} = A_\rho + B_\omega + A_i + A_j + B_i + B_j - V_{ij} \text{ and}$$

$$M_{ji} = A_\rho + B_\omega + A_i + A_j + B_i + B_j - V_{ji},$$

where $V_{ij} = \min(\min(A_j, B_i), A_i + A_j - Q_{AB}, B_i + B_j - Q_{BA})$,
 $V_{ji} = \min(\min(A_i, B_j), A_i + A_j - Q_{AB}, B_i + B_j - Q_{BA})$, and
 $A_\rho = \sum_{k \in \rho} A_k$, $B_\omega = \sum_{k \in \omega} B_k$, and Q_{AB} (Q_{BA}) is the additional time after A_ρ (B_ω)
 necessary to complete the processing of the jobs of $\rho(\omega)$ on $B(A)$ under π_{ij} and π_{ji} ($\underline{\pi}_{ij}$
 and $\underline{\pi}_{ji}$).

Proof. The makespan of a given job sequence in m -machine flow shops can be represented by the length of the critical (longest) path in an acyclic network [9, p. 131]. From the definition of A_ρ , B_ω , Q_{AB} and Q_{BA} , we have $M_{ij} = A_\rho + R_{ij} + B_\omega$, where R_{ij} is the length of the critical path in the “activity-on-arc” PERT network in Figure 1a. Since $\max(A_i + A_j + B_j, A_i + B_i + B_j) = A_i + A_j - \min(A_j, B_i) + B_i + B_j$, the networks in Figures 1a and 1b have the same critical path length. It suffices now to extract the term $A_i + A_j + B_i + B_j$ from R_{ij} and observe that $R_{ij} = A_i + A_j + B_i + B_j - V_{ij}$, which verifies the representation of M_{ij} . When we switch i and j , one obtains the representation of M_{ji} . ■

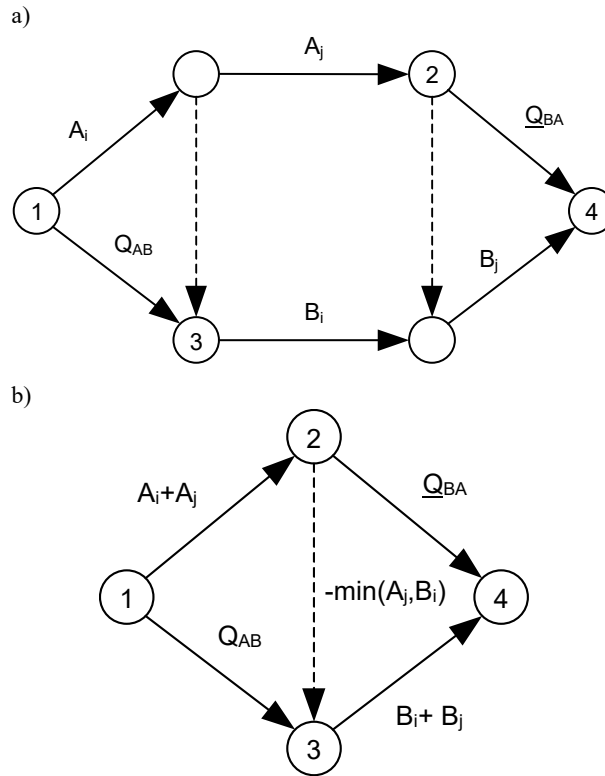


Fig. 1. PERT networks for computing R_{ij}

2.2. Optimal sequencing rules

When the job processing times are deterministic, the proposed representations of M_{ij} and M_{ji} illustrate the nature of simplicity of the well-known Johnson's sequencing rule [4]:

$$\text{job } i \text{ precedes job } j \text{ if } \min(A_j, B_i) > \min(A_i, B_j). \quad (1)$$

Since V_{ij} and V_{ji} defined in Theorem 1 differ only in the terms $\min(A_j, B_i)$ and $\min(A_i, B_j)$, the inequality $\min(A_j, B_i) \geq \min(A_i, B_j)$ implies $V_{ij} \geq V_{ji}$, and consequently, $\min(A_j, B_i) \geq \min(A_i, B_j)$ implies $M_{ij} \leq M_{ji}$. Since $\min(A_j, B_i) \geq \min(A_i, B_j)$ and $\min(A_k, B_j) \geq \min(A_j, B_k)$ imply $\min(A_k, B_i) \geq \min(A_i, B_k)$ [4], any job sequence determined by (1) is optimal. The term $\min(A_j, B_i)$ can be interpreted as the time savings from allowing i and j to overlap under (i, j) [6]. Clearly, if these two jobs are not allowed to overlap, their processing time would be $A_i + B_i + A_j + B_j$. Thus, Johnson's rule (1) states that job i must precede job j if the time savings under (i, j) are greater than under (j, i) .

Unfortunately, the above simple arguments are no longer valid when the job processing times A_k and B_k are independent random variables. The following example shows that $E(\min(A_j, B_i)) \geq E(\min(A_i, B_j))$ does not imply $E(M_{ij}) \leq E(M_{ji})$.

Example 1. Consider three jobs 1, 2 and 3, and assume 1 and 3 have deterministic processing times $P(A_1 = 1) = P(B_1 = 2) = P(A_3 = 2) = P(B_3 = 1) = 1$, while A_2 and B_2 are independent exponential random variables with a mean of 1. For the job sequences $\pi_{12} = (1, 2, 3)$ and $\pi_{21} = (2, 1, 3)$, we have $E(\min(A_2, B_1)) = 1 - e^{-2} > 1 - e^{-1} = E(\min(A_1, B_2))$, but $E(M_{12}) = 5.5 + \frac{1}{2} e^{-4} \approx 5.5092 > 5.3679 \approx 5 + e^{-1} = E(M_{21})$.

Recall that a random variable X is stochastically smaller than another random variable Y , written $X \leq_{\text{st}} Y$, if $P(X \leq t) \geq P(Y \leq t)$ for every t . Moreover, $X \leq_{\text{st}} Y$ implies $E(X) \leq E(Y)$. Below $[X|Y = y]$ denotes a conditional random variable.

From Theorem 1 one obtains immediately the following result.

Theorem 2 [8]. *A sufficient condition for $M_{ij} \leq_{\text{st}} M_{ji}$ is*

$$[\min(A_j, B_i)|A_i + A_j = a, B_i + B_j = b] \geq_{\text{st}} [\min(A_i, B_j)|A_i + A_j = a, B_i + B_j = b]$$

for all a and b in the supports of $A_i + A_j$ and $B_i + B_j$, respectively.

A non-negative random variable X has the Gompertz distribution (see, e.g., [3, p. 25]) with parameters $\lambda > 0$ and $\gamma > 0$, written $X \sim G(\lambda, \gamma)$, if its survival function is given by

$$S(x) = P(X > x) = \exp[-\lambda(e^{\gamma x} - 1)/\gamma] \text{ for } x \geq 0. \quad (2)$$

The median of X is

$$m = \ln(1 + \gamma \ln 2 / \lambda) / \gamma. \quad (3)$$

From (3) we have $\lambda = \gamma \ln 2 / (e^{\gamma m} - 1)$, and after substituting this expression to (2), one obtains the equivalent form of the survival function:

$$S(x) = 2^{-\frac{(e^{x\gamma} - 1)(e^{\gamma m} - 1)}{\gamma m}} \text{ for } x \geq 0. \quad (4)$$

Since $\lim_{\gamma \rightarrow 0} S(x) = \exp(-\lambda x) = \exp(-x \ln 2 / m)$, the exponential distribution with the shape parameter λ (or equivalently with the median $m = \ln 2 / \lambda$) is the limiting distribution of the Gompertz distribution when γ approaches zero. On the other hand, using (4), we have $\lim_{\gamma \rightarrow \infty} S(x) = 1$ for $x < m$ and $\lim_{\gamma \rightarrow \infty} S(x) = 0$ for $x > m$, and thus the Gompertz distribution becomes degenerate at m , when γ approaches infinity.

Theorem 3 [5]. *Let the job processing times A_k and B_k be independent random variables such that $A_k \sim G(\alpha_k, \gamma)$ and $B_k \sim G(\beta_k, \gamma)$ for $k = 1, 2, \dots, n$. Then a sufficient condition for $M_{ij} \leq_{st} M_{ji}$ is $\alpha_j + \beta_i \leq \alpha_i + \beta_j$.*

Theorem 3 leads directly to the following result.

Corollary 1 [5]. *Let the job processing times A_k and B_k be independent random variables such that $A_k \sim G(\alpha_k, \gamma)$ and $B_k \sim G(\beta_k, \gamma)$ for $k = 1, 2, \dots, n$. Then the rule: job i precedes job j if $\alpha_j + \beta_i < \alpha_i + \beta_j$ yields a job sequence that minimizes stochastically the makespan.*

Since the degenerate and exponential distributions are limiting distributions of the Gompertz distribution, the sequencing rule stated in Corollary 1 can be regarded as a generalization of Johnson's and Talwar's sequencing rules.

Corollary 2

(i) [4] *Let the job processing times A_k and B_k be deterministic for $k = 1, 2, \dots, n$. Then the rule: job i precedes job j if $\min(A_j, B_i) > \min(A_i, B_j)$ yields a job sequence that minimizes the makespan.*

(ii) [1, 11] *Let the job processing times A_k and B_k be independent and exponential random variables with shape parameters α_k and β_k , respectively, for $k = 1, 2, \dots, n$. Then the rule: job i precedes job j if $\alpha_j + \beta_i < \alpha_i + \beta_j$ yields a job sequence that minimizes stochastically the makespan.*

3. Three-machine flow shops

Although all results in this section can be generalized to the case of an arbitrary number of machines, for clarity of presentation, we have decided to discuss the case of three machines.

A set of jobs, $\{1, 2, \dots, n\}$, available at time zero has to be processed in a flow shop with three machines A, B and C, that is, each job has to be processed first on A, next on B, and next on C. The problem is to determine a job sequence with the minimum makespan, and this problem is known to be strongly *NP*-hard [2].

3.1. Makespan representations

Let A_k , B_k , and C_k denote the processing times of job k on machine A, B and C, respectively. As Stefan observed, when the jobs of a given sequence go through a flow shop in the reverse order and in the opposite direction (first on C, next on B, and last on A), then the makespans of both sequences are the same. As previously let M_{ij} (M_{ji}) be the makespan of $\pi_{ij} = (\rho, i, j, \omega)$ and $\underline{\pi}_{ij}$ ($\pi_{ji} = (\rho, j, i, \omega)$ and $\underline{\pi}_{ji}$).

The following theorem shows convenient representations of M_{ij} and M_{ji} .

Theorem 4 [7]. *The makespans M_{ij} and M_{ji} can be represented as follows:*

$$M_{ij} = A_\rho + C_\omega + A_{ij} + B_{ij} + C_{ij} - V_{ij} \quad \text{and} \quad M_{ji} = A_\rho + C_\omega + A_{ij} + B_{ij} + C_{ij} - V_{ji},$$

where V_{ij} and V_{ji} are the shortest path lengths in the networks of Figure 2, and

$$A_\rho = 3 \sum_{k \in \rho} A_k, \quad C_\omega = 3 \sum_{k \in \omega} C_k, \quad A_{ij} = A_i + A_j, \quad B_{ij} = B_i + B_j, \quad C_{ij} = C_i + C_j, \quad \text{and}$$

Q_{AB} (Q_{AC}) = the additional time after A_ρ necessary to complete the processing of the jobs of ρ on B (C) under π_{ij} and π_{ji} ,

Q_{CB} (Q_{CA}) = the additional time after C_ω necessary to complete the processing of the jobs of ω on B

and (A) under $\underline{\pi}_{ij}$ and $\underline{\pi}_{ji}$.

Below we present some interpretations for the case of π_{ij} and $\underline{\pi}_{ij}$; analogous interpretations are valid for the case of π_{ji} and $\underline{\pi}_{ji}$.

Let $S_{ij} = \min(A_j + B_j, B_i + C_i, A_j + C_i)$, that is, S_{ij} is the shortest path length in the subnetwork restricted to heavy arcs in the network representing V_{ij} . Then $A_{ij} + B_{ij} + C_{ij} - S_{ij}$ is the makespan of (i, j) . S_{ij} can be interpreted as the time savings from allowing jobs i and j to overlap on A, B and C under (i, j) . Clearly, if i and j are the only jobs in the system, and they are not allowed to overlap, then their processing time would be $A_{ij} + B_{ij} + C_{ij}$. The terms $\min(A_j, B_i)$ and $\min(B_j, C_i)$ can be interpreted as the time savings from allowing jobs i and j to overlap on A and B, and B and C under (i, j) .

From Theorem 4 one can easily derive the following result that is due to Włodzimierz Szwarz, a supervisor of Stefan's master thesis.

Corollary 3 [10]. *For the deterministic job processing times, sufficient and equivalent conditions for $M_{ij} \leq M_{ji}$ are*

- (i) $\min(A_j, B_i) \geq \min(A_i, B_j)$, $\min(B_j, C_i) \geq \min(B_i, C_j)$, and $S_{ij} \geq S_{ji}$.
- (ii) $\min(A_j, B_i) \geq \min(A_i, B_j)$, $\min(B_j, C_i) \geq \min(B_i, C_j)$, and $\min(A_j, C_i) \geq \min(A_i, C_j)$.

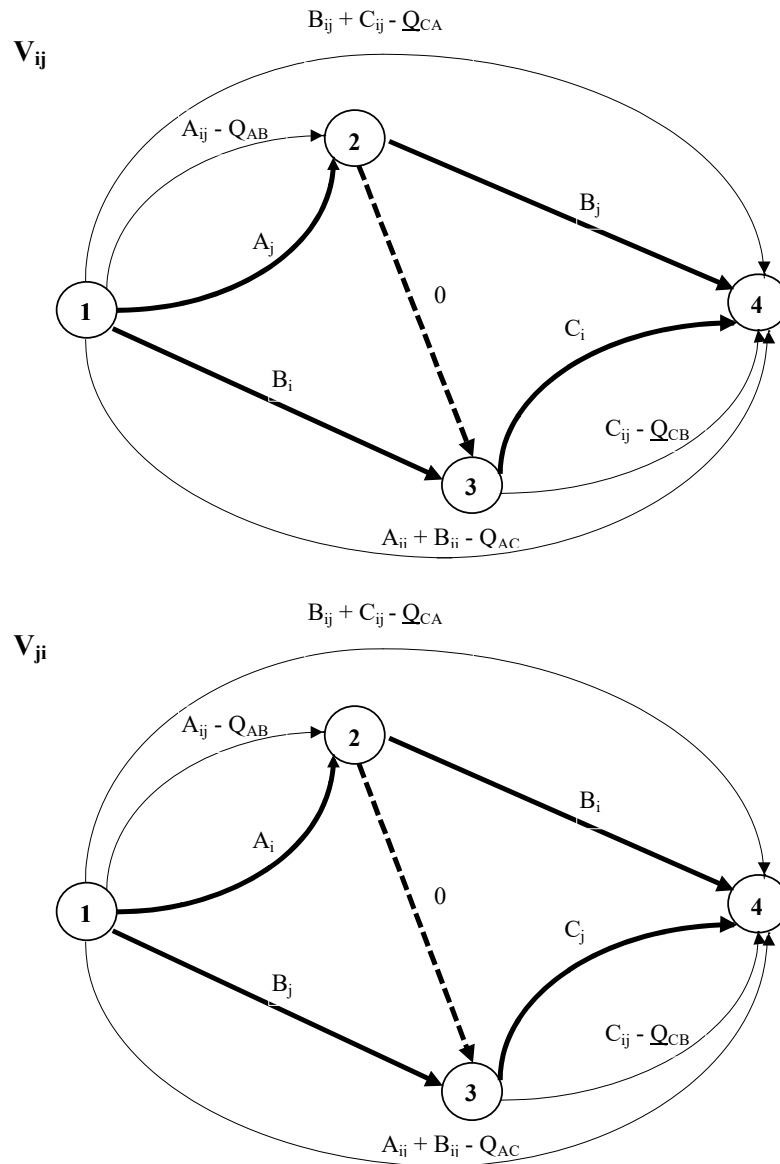


Fig. 2. Shortest path networks for computing V_{ij} and V_{ji}

3.2. Optimal sequencing rules

Assume that the job processing times A_k , B_k and C_k are independent random variables. The following theorem can be regarded as a stochastic counterpart of Corollary 3 (ii).

Theorem 5 [7]. *A Sufficient condition for $M_{ij} \leq_{st} M_{ji}$ is*

$[\min(A_j, B_i)/A_i + A_j = a, B_i + B_j = b] \geq_{st} [\min(A_i, B_j)/A_i + A_j = a, B_i + B_j = b]$, and
 $[\min(B_j, C_i)/B_i + B_j = b, C_i + C_j = c] \geq_{st} [\min(B_i, C_j)/B_i + B_j = b, C_i + C_j = c]$, and
 $[\min(A_j, C_i)/A_i + A_j = a, C_i + C_j = c] \geq_{st} [\min(A_i, C_j)/A_i + A_j = a, C_i + C_j = c]$,
 for all a, b and c in the supports of $A_i + A_j$, $B_i + B_j$ and $C_i + C_j$, respectively.

Using Theorem 5 one can prove the next result concerning Gompertz distributed job processing times.

Theorem 6 [5]. *Let the job processing times A_k , B_k and C_k be independent random variables such that $A_k \sim G(\alpha_k, \gamma)$, $B_k \sim G(\beta_k, \gamma)$, and $C_k \sim G(\chi_k, \gamma)$ for $k = 1, 2, \dots, n$. Then a sufficient condition for $M_{ij} \leq_{st} M_{ji}$ is $\alpha_j + \beta_i \leq \alpha_i + \beta_j$ and $\beta_j + \chi_i \leq \beta_i + \chi_j$.*

Under the assumption stated in Theorem 6, we propose the following algorithm whose idea is adopted from [10].

Algorithm 1

Step 1. Determine a sequence π by applying the rule: i precedes j if $\alpha_j + \beta_i < \alpha_i + \beta_j$.

Step 2. Reorder all segments of π according to: i precedes j if $\beta_j + \chi_i < \beta_i + \chi_j$. (A subsequence η of π is called a segment if $\alpha_j + \beta_i = \alpha_i + \beta_j$ for all jobs i and j belonging to η .)

Corollary 4 [5]. *Let the job processing times A_k , B_k and C_k be independent random variables such that $A_k \sim G(\alpha_k, \gamma)$, $B_k \sim G(\beta_k, \gamma)$, and $C_k \sim G(\chi_k, \gamma)$ for $k = 1, 2, \dots, n$. If $(\alpha_j + \beta_i - \alpha_i - \beta_j)(\beta_j + \chi_i - \beta_i - \chi_j) \geq 0$ for all $i \neq j$, then Algorithm 1 finds a job sequence that minimizes stochastically the makespan.*

Corollary 4 can be regarded as a generalization of the results in [7, 10] that dealt with the exponential and deterministic cases.

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O nieznanym przyczynku Stefana Chanasa do analizy stochastycznych systemów przepływowych

W artykule rozważany jest problem wyznaczania optymalnej kolejności obsługi zadań w stochastycznych systemach przepływowych. Za kryterium optymalizacji przyjmuje się minimalizację czasu zakończenia obsługi. Zaprezentowano ostatnie osiągnięcia w analizie stochastycznych systemów przepływowych, w których tzw. zwrotne właściwości odgrywają kluczową rolę. Zwrotność to następująca własność każdego systemu przepływowego: jeżeli zadania są obsługiwane w odwrotnej kolejności i w odwrotnym porządku, to czas zakończenia obsługi pozostaje taki sam.

Artykuł poświęcony jest pamięci Stefana Chanasa, mojego długoletniego przyjaciela i faktycznego promotora mojego doktoratu. To On zauważył pierwszy, że własność zwrotności musi być wykorzystana w analizie złożoności systemów przepływowych.